TERMINAL DYNAMICS APPROACH TO DISCRETE EVENT SYSTEMS

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ABSTRACT

This paper presents and discusses a mathematical formalism for simulation discrete event dynamic (DED)-a special type. of "mm-made" systems to serve specific purposes of information processing. The main objective of this work is to demonstrate that the mathematical formalism for DED can be based upon terminal model of Newtonian dynamics which allows one to relax Lipschitz conditions at some discrete points.

1. INTRODUCTION

Complexity of dynamical system performance can be significantly enriched by exploiting a terminal model of nonlinear dynamics introduced and discussed in [3,4]. As shown there, this model can capture stochastic properties of information processing without utilizing a random number generators: a multi-choice response to a deterministic massage is provided by a failure of uniqueness of the solution due to relaxation of Lipschitz conditions at some discrete points. However, since terminal dynamics is differentiable almost everywhere (excluding these discrete points), it preserves most of the analytical structure of mathematical formalism of classical theory of differential equations. A combination of such '(cent radictory" properties - the analiti ci ty and discreetness - gives motivation to simulate discrete event systems using terminal dynamics.

Discrete event dynamics represents a special type of "man-made") systems to serve specific purposes of information processing [1]. Models of such systems fall int o several broad categories of various levels of abstraction. At the logical level, one is only concerned with the logical order of the events, i.e., with qualitative, or structural behavior of a system; these models are usually deterministic and untimed. At the performance level, the models involve random event lifetimes and other probabilistic entities, and are primary concerned with charact erizing the quantitative aspects of the system. The performance models are usually based upon generalized semi-Markov processes and st ochast ic Petri nets.

This paper presents and discusses a mathematical formalism for discrete events systems based upon terminal model of Newtonian dynamics.

2. TIME-DRIVEN DISCRETE SYSTEMS

A broad class of complex dynamical behaviors can be derived from a simple differential equation [3]:

$$\dot{x} = x^{1/3} \sin \omega t, \omega = \text{cost} \qquad (1)$$

The solution to Eq. (1) can be presented in a closed form. Indeed, assuming that $x \to O$ at t = O, one obtains a regular solution:

$$x = \pm(\frac{4}{3\omega} \sin^2 \frac{\omega}{2} t)^{3/2} \quad \text{if } x \neq 0 \qquad (2)$$

and a singular solution (an equilibrium point):

$$x = 0 \qquad (3)$$

Clearly, the Lipzschitz condition at the quilibrium point x= O fails since

$$\left| \frac{d\dot{x}}{dx} \right| = \frac{1}{3} x^{-\frac{2}{3}} \sin \omega t \to \infty \text{ at } x \to 0. \qquad (4)$$

As follows from (2), two different solutions are possible for "almost the. same" initial conditions. The fundamental propert y of this result is that the di vergence of these solutions from $x = 0$ is characterized by an unbounded para meter, σ:

$$\sigma = \lim_{t \to 0} \left[\frac{1}{t} \ell n \frac{(\frac{4}{3\omega} \sin^2 \frac{\omega}{2} t)^{3/2}}{2 \mid x_o \mid} \right| = \infty, \quad \mid x_o \mid \to 0 \qquad (5)$$

where t_o is an arbitrarily small (but finite) positive quantity. The rate of divergence (5) can be defined in an arbitrarily small time intend, because the initial infinitesimal distance between the solutions *(2)* becomes finite during the small interval t_o. One should recall that in the classical case when the Lipschit z condit ion is sati sfied, the dist ance bet ween two diverging solutions can become finite only at $t \to \infty$ if initially this distance was infinitesimal.

The solution (2) and (3) co-exist at t = O, and that is possible because at this point the Lipschitz condition fails (see Eq. 4).

Since:

$$\frac{\partial \dot{x}}{\partial x} > 0 \quad \text{at} \quad |x| \neq 0, \quad t > 0, \tag{6}$$

the singular solution (3) is unstable, and it departs from rest following Eq. (3). This solution has two (positive and negative) branches, and each branch can be chosen with the same probability 1/2. It should be noticed that as a result of (4), the motion of the particle can be initiated by infinitesimal disturbances (that never can occur when the Lipschitz condition is in place: an infinitesimal initial disturbance cannot become finite in finite time).

Strictly speaking, the solution (2) is valid only in the time interval

$$0 \leq t \leq \frac{2\pi}{\omega}, \tag{7}$$

and at $t = \frac{2\pi}{\omega}$ it coincides with the singular solution (3). For $t > 2\pi/\omega$, Eq. (2) becomes unstable, and the motion repeats itself to the accuracy of the sign in Eq. (2).

Hence, the solution performs oscillations with respect to its zero value in such a way that the positive and negative branches of the solution (2) alternate randomly after each period equial to $2\pi/\omega$.

Let us introduce another variable:

$$\dot{y} = x, \quad (y = 0 \text{ at } x = 0). \tag{8}$$

After the first time interval $t = \frac{2\pi}{\omega}$

$$y = \pm \int_0^{2\pi/\omega} \left(\frac{4}{3\omega} \sin^2 \frac{\omega}{2} t \right)^{3/2} dt = (64(3\omega)^{-5/2} = \pm h \tag{9}$$

After the second time interval $t = \frac{4\pi}{\omega}$

$$y = \pm h \pm h \tag{10}$$

etc.

obviously, the variable y performs an unrestricted symmetric random walk: after each time period $\tau = 2\pi/\omega$ it changes its value on $\pm h$. The probability $f(y,t)$ is governed by the following difference equation:

$$f(y,t+\frac{2\pi}{\omega}) = \frac{1}{2}f(y-h,t) + \frac{1}{2}f(y+h,t), \int_{-\infty}^{\infty} f(y,t)dy = 1, \tag{11}$$

where h is expressed by Eq. (9).

Eq. (11) defines f as a function of two discrete arguments:

$$y = \pm kh, \text{ and } t = l\tau, \ \tau = \frac{2\pi}{\omega}, \ k,l = 0,1,2,\cdots \text{ etc.} \tag{12}$$

For convenience, we will keep for discrete variable y and t the same notations as for their continuous versions.

By change of the variables:

$$z = \varphi(y), y = \text{y-}^{1}(z) \tag{13}$$

one can obtain a stochastic process with a prescribed probability distribution:

$$\psi(z,t) = f[\varphi^{-1}(z),t] \left| \frac{d\varphi^{-1}}{dz} \right| \tag{14}$$

implemented by the dynamical system (1), (8), and (13).

Here z is also considered as a discrete variable changing at each time-step τ according to Eq. (12) and (13).

It is important to emphasize that the system (1), (8), (13) does not have any random input: the randomness is "generated" by the differential operator in (1) due to violation of the Lipschitz condition.

Let us modify Eq. (1) as following:

$$\dot{u} = \varphi(y)u^{1/3} \sin \omega t, \ \varphi(y) = z_1 > 0 \tag{15}$$

where y is defined by Eq. (8).

The solution to Eq. (15) can be written in the form similar to (2):

$$u = {}^{*}\left[-\frac{4\varphi(h)}{3\omega}\sin^2\frac{\omega}{2}t\right]^{3/2} \tag{16}$$

where h is given by Eq. (9), while the probability density of $\varphi(y, t)$ is expressed by Eq. (14).

Now we are ready to introduce a dynamical model for the generalized random walk:

$$\dot{v} = \alpha u, \quad \alpha = \frac{(3\omega)^{5/2}}{64\sqrt{\varphi(h)}} = \text{const} \tag{17}$$

Indeed, after the first time interval $t = \frac{2\pi}{\omega}$

$$v = \pm\alpha\int_0^{2\pi/\omega}\left[\frac{4\varphi(h)}{3\omega}\sin^2\frac{\omega}{2}t\right]^{3/2}dt = \pm\varphi(h), \tag{18}$$

After the third time interval $t = \frac{4\pi}{\omega}$

$$v = \pm\varphi(h) \pm \varphi(h) \text{ etc..} \tag{19}$$

Thus, the variable v performs a symmetric generalized random walk: after each time period $\tau = 2\pi/\omega$, it changes it value on $\pm\varphi(h)$. But $\varphi(h)$, in turn, is also a random variable: its probability density follows from Eqs. (13) and (14). Hence, at each step the variable v have a probability p_k to move from any point v_0 to an arbitrarily selected point v_k.

It should be emphasized that this generalized random walk is implemented by the dynamical system (1), (8), (14), (15), and (18). Considering v as a discrete variable at $t =$ 117-

$$v = \pm n\varphi(h), \tag{20}$$

one obtains the governing difference equation for the probability distribution of v:

$$\Phi(v, t + \frac{2\pi}{\omega}) = \frac{1}{2}\sum_{k=1}^{n} p_k \{\Phi[v - \varphi(kh)] + \Phi[v + \varphi(kh)]\} \tag{21}$$

$$\int_{-\infty}^{\infty} \Phi(v,t)dv = 1 \tag{22}$$

where

$$p_k = f[\varphi^{-1}(kh,\ t)]\ |\ \frac{d\varphi^{-1}(kh,t)}{dz}\ | \tag{23}$$

and n is the number of the discrete values in (12).

Applying the terminology of Discrete Event Systems, both simple and generalized dynamical models of random walks can be characterized as time-drivem systems, since here with every "clock tick" the state is expected to change with a prescribed probability.

We will stress that although the dynamical system (l), (8), (14), (15), and (18) has random solution, this randomness is wall-organized: it can be uniquely described in probability terms with the aid of Eq. (21). Indeed, if the initial probability distribution

$$\Phi(v,\ o) = \Phi_0(v) \tag{24}$$

$$f(y, 0) = f_0(y) \tag{25}$$

$$y = \varphi(z) \tag{26}$$

are given, then all the variables in the right-hand part of Eq. (21) can be computed at each time step τ, and that uniquely defines the left-hand part of Eq. (21), and therefore, the evolution of the probability distribution $\Phi(v,t)$.

Based upon the? system (l), (8), (14), (15), and (18), one can find the functions $\Phi_0(v)$, $f_0(y)$ and $\varphi(z)$ in Eqs. (24), (25) and (26), respectively, such that they provide a required evolution of the probability distribution $\Phi(v,t)$.

So far we were concerned with the performance aspects of the time-driven dynamics. On the logical level, the only function $\varphi(z)$ in Eq. (24) contributes into the logical structure of the system: it prescribes the probabilities p_k in Eq. (21) that the solution move from any point v_0 to an arbitrarily selected point v_k.

3. EVENT-DRIVEN DISCRETE SYSTEMS

Let us return to Eq. (1) and assume that it is driven by a vanishingly small input ε:

$$\dot{x} = x^{1/3} \sin \omega t + \varepsilon, \quad \varepsilon \to 0 \tag{27}$$

From the viewpoint of information processing, this input can be considered as a massage, or an event. This massage can be ignored when $\dot{x} \neq 0$, or when $\dot{x} = 0$, but the system is stable, i.e., $x = \pi\omega, 2\pi\omega, \ldots$ etc. However, it becomes significant during the instants of instability when $\dot{x} = 0$ at $t = 0, \pi/2\omega, \ldots$ etc. Indeed, at these instants, the solution to (27) would have a choice to be positive or negative if $\varepsilon = 0$, (see Eq. (2)). However, with $\varepsilon \neq 0$

$$\operatorname{sgn} x = \operatorname{sgn} \dot{x} \quad \text{at } t = 0, \pi/2\omega, \ldots \text{ etc.} \tag{2s}$$

i.e. the sign of $\dot{x}$ at the critical instances of time (28) uniquely defines the evolution of the dynamical system (27).

Actually the event ε may represent an output of a microsystem which uniquely controls the behavior of the original dynamical system (27).

The solution to Eq. (8) now becomes deterministic. if $\operatorname{sgn} \varepsilon \neq 0$ at critical points (28), and instead of Eqs. (9), and (10), one obtains, respectively:

$$y = \operatorname{sgn} \varepsilon \int_0^{2\pi/\omega} \left(\frac{4}{3\omega} \sin^2 \frac{\omega}{2} t \right)^{3/2} dt = h \operatorname{sgn} \varepsilon, \tag{29}$$

$$y = h \operatorname{sgn} \varepsilon_1 + h \operatorname{sgn} \varepsilon_2 + \text{etc.} \tag{30}$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are the values of ε at $t = O, 2\pi/\omega$ etc., respectively.

The probability y $f(y, t)$, instead of Eq. (11), is governed by the following difference equation:

$$f(y, t + \frac{2\pi}{\omega}) = pf(y - h, t) + (1 - p)f(y + h, t) \tag{31}$$

where

$$p = \begin{cases} 1 & \text{if } sgn\ \varepsilon = 1 \\ 0 & \text{if } \operatorname{sgn}\ \varepsilon = -1 \\ \frac{1}{2} & \text{if } \varepsilon = 0 \end{cases} \tag{32}$$

Actually the evolution of the probability distribution in Eq. (31) is represented by rigid shifts of the initial probability distribution $f(y, O)$, unless $\operatorname{sgn} \dot{x} = 0$.

Let us modify now Eq. (15) in the same way as Eq. (l):

$$\dot{u} = \varphi(y) u^{1/3} \sin \omega t + \varepsilon, \quad \varepsilon \to 0 \tag{33}$$

Then the solution to Eq. (17), is written as:

$$v = \$7(\}1)\ \operatorname{sgn}\varepsilon_1 + \varphi(h)\operatorname{sgn}\varepsilon_2 + \text{“ “ “etc.} \tag{34}$$

where $\varepsilon_1, \varepsilon_2, \cdots$ are the values of ε at t = O, $2\pi/\omega$ etc., respectively.

Unlike the previous case (see Eqs. (18-19)), now the variable v defined by Eqs. (l), (8), (17), and (33) perform a non-symmetric generalized random walk: after each time period $\tau = 2\pi/\omega$, it changes its value on $\varphi(h)\operatorname{sgn}\varepsilon$, i.e., in a certain direction defined by the event ε, (unless sgn ε = O). But $\varphi(h)$ is still a random variable whose probability density follows from Eqs. (13) and (14). Hence, at each step the variable v have a probability y to move from any point v_0 to an arbitrary selected poi nt v_k in the direction defined by sgn &.

Introducing v by Eq. (34), one obtains the governing difference equations for the probabilit y dist ribut ion of v:

$$\Phi(v, t + \frac{2\pi}{\omega}) = \sum_{k=1}^{n} p_k \{p\Phi[v - \varphi(kh)] + (1-p)\Phi[v + \varphi(kh)]\} \tag{35}$$

$$\int_{-\infty}^{\infty} \varphi(v,t)dv = 1 \tag{36}$$

where p_k and p are expressed by Eqs. (23) and (32), respect ively.

Thus, although the dynamical system (1), (8), (17), and (33) has random solution, its probabilistic properties are uniquely defined by Eqs. (35) and the constraint (36).

It should be noticed that the solution to Eqs. (35) or (21) does always automatically satisfy t he constraint (36). That is why, in general, it is more convenient to consider Φ in Eqs. (35) or (21), as an auxiliary (not normalized) function while introducing the. normalized probability distribution by the following formula:

$$\hat{\Phi}(v,t) = \frac{\Phi(v,t)}{\int_{-\infty}^{\infty}\Phi(v,t)dv} \tag{37}$$

Then

$$\int_{-\infty}^{\infty} \hat{\Phi}(v,t)dv \equiv 1 \tag{38}$$

Comparing the previously considered dynamical system (1), (8), (14), (15), and (1S) with the dynamical system (1), (8), (17), and (33), one can see a funds.ment al difference: the latter is driven by a massage ε whose changes are independent upon the "clock tick" of the dynamical system itself. This massage, in general, can be an output of another dynamical system with its own time scale, or it can depend upon the variables (or their probabilities) of the original dynamical system. In all these cases, the massage ε can be treated as an independent event, and therefore, the dynamical system (1), (8), (17), and (33) is driven by events.

In conclusion of this section we will review the structure of the solution to Eq. (35) which describes the evolution of the probability y distribut ion $\Phi(v,t)$.

Introducing the displacement operators

$$E_t\theta(t) = \theta(t + \frac{2\pi}{\omega}),\ E_v\theta(v) = \theta(v + h) \tag{39}$$

one can rewrite Eq. *(35)* in the following form:

$$\left\{E_t - \sum_{k=1}^{n} p_k [pE^{-k} + (1-p)E^k]\right\}\Phi(v,t) = O \tag{40}$$

Applying Bool's symbolic method, i.e. replacing the operator E_v *by* a constant λ, one arrives at an ordinary difference equation:

$$(E_t - \tilde{\lambda})\Phi = O,\quad \tilde{\lambda} = \sum_{k=1}^{n} p_k [p\lambda^{-k} + (1-p)\lambda^k] = \text{const} \tag{41}$$

The solution to this equation in a symbolic form is:

$$\Phi(v,t) = \tilde{\lambda}^{\ell}\varphi(v) \tag{42}$$

where $\varphi(v) = \Phi(v, O)$ is the initial probability distribution of v, and $\ell = 0, 1, 2, \cdots$ etc. (see Eq. (14)).

Obviously that for $\ell = O$:

$$\Phi(v,t) = \varphi(v) \tag{43}$$

Then, for $\ell = 1$:

$$\Phi(v, t_1) = \sum_{k=1}^{n} p_k \left[p\varphi(v - \frac{2\pi k}{\omega}) + (1-p)\varphi(v - \frac{2\pi k}{\omega}) \right] \tag{44}$$

Continuing this process for ℓ = 2,3,. . . etc., one arrives at the following recurrent relationships:

$$\Phi(v, t_{\ell+1}) = \sum_{k=1}^{n} p_k \left[p\Phi(v - \frac{2\pi k}{\omega}, t_\ell) + (1-p)\Phi(v + \frac{2\pi k}{\omega}, t_\ell) \right] \tag{45}$$

Hence, based upon the initial condition (43), the evolution of the probability distribution $\Phi(v, t)$ can be uniquely defined by Eq. (45).

The solution to Eq. (21) can be obtained from (45) if one set p = 1/2.

It should be emphasized that Eq. (49) is a piecewise linear: indeed, it is linear as long as the massage ε does not change its sign. However, since this sign may depend upon time t, or upon the st ate variable v, or even upon its probabilit y j(v), globally Eq. (49) can be linear with variable coefficients, or even nonlinear.

4. SYSTEMS DRIVEN BY TEMPORAL EVENTS

In this section we will assume that the message ε is given as a function of time :

$$\varepsilon = \varepsilon(t) \tag{46}$$

We will start with a simplified version of the system (l), (8), (17), (33), assuming that in Eq. (14)

$$y = \begin{cases} z \text{ if } |z| \leq nh \\ \text{O otherwise} \end{cases} \tag{47}$$

This means that at each state the variable v have equal probability to have steps $\pm h, \pm 2h, \text{“ “ “} \pm nh$.

One can rewrite the system in the following form:

$$\dot{u} = yu^{1/3} \sin \omega t + \varepsilon(t), \ | \ \varepsilon(t) \ | \ll | \ u \ | \tag{48}$$

$$'i) = \alpha u, \ \ \alpha = \text{ const } \text{ (see Eq.(17))} \tag{49}$$

As shown above, Eq. (11) can be implemen ted by the dynamical system (l), (8) which is not coupled with Eqs. (4S) and (49).

Now Eq.(35) describing thee evolution of the probability distribution of the variable v reduces to:

$$\Phi(v, t + \frac{2\pi}{\omega}) = \frac{1}{n} \sum_{k=1}^{n} p\Phi(v - kh) + (1 - p)\Phi(v + kh) \tag{50}$$

while p is given Eq. (35).

We will illustrate the solution to the dynamical system (48) as well as to the associated probability equation (50) by assuming that in Eq. (48):

$$\varepsilon(t) = .50 \sin \Omega t, \ \ \varepsilon_0 = \text{ const } << 1 \tag{51}$$

First consider the case when ω/Ω is irrational, i.e.,

$$\frac{\omega}{\Omega} \neq \frac{m}{n} \tag{52}$$

where m and n are integers.

Then at the critical points:

$$t = \frac{2\pi}{\omega}\ell, \ \ell = 1, 2 \cdots \tag{53}$$

$$E(t) \neq \text{ O} \tag{54}$$

and therefore, in Eq. (47):

$$\tilde{\lambda} = \begin{cases} \frac{1}{n}\sum_{k=1}^{n}\lambda^{k} & \text{if } 1-\frac{1}{2\ell} \le \frac{\Omega}{\omega} \le 1+\frac{1}{2\ell} \\ \frac{1}{n}\sum_{k=1}^{n}\lambda^{-k} & \text{if } 1 \le \frac{\Omega}{\omega} \le 1+\frac{1}{2\ell} \end{cases} \qquad (55, 56)$$

For $n = 1$ the solution to Eq. (50) can be presented in the form of a propagating wave:

$$\Phi = \lambda^{t}\varphi(v) = E_{v}^{t}\varphi(v) = \varphi(v \pm t) \qquad (57)$$

where the signs + and - correspond to (55) and (W), respectivel y.

Actually in this particular case the solution remains fully deterministic if the initial conditions to the dynamical system (48), (49) are deterministic.. In terms of the solution (72), this would mean that $\varphi(v)$ as well as $\varphi(v \pm t)$ are the δ functions.

Turning to the general case of Eq. (50), let us apply the solution (45). Then, for the first time-step:

$$\Phi(v, t_1) = \frac{1}{n}\sum_{k=1}^{n}\varphi(v - \frac{2\pi k}{\omega}), \quad \varphi = \Phi(v, 0) \qquad (5s)$$

Hence, the solution starts with n waves propagating in the same direction, but with different speeds. Application of the solution (45) t o the next time-steps shows that these waves start interacting and the structure of the solution beco mes as complex as the linear wave interference.

For the case (56), instead of (58) one obtains

$$\Phi(v, t_1) = \frac{1}{n}\sum_{k=1}^{n}\varphi(v + \frac{2\pi k}{\omega}), \quad \varphi = \Phi(v, 0) \qquad (59)$$

i.e., a similar wave train propagates in opposite direction.

Let us replace the condition (52) by the following:

$$\frac{\omega}{\Omega} = \frac{m}{n} \qquad (60)$$

Then at some of the critical points (53):

$$\operatorname{sgn} \varepsilon(t) = o \tag{61}$$

and therefore, all the three cases in (32) can occur.

Then, instead of (58) and (59), the following solution can be obtained for the first time-step:

$$\Phi(v, t_1) = \frac{1}{2n} \sum_{k=1}^{n} \left[\varphi(v - \frac{2\pi k}{\omega}), + \varphi(v + \frac{2\pi k}{\omega}) \right], \varphi = \Phi(v, 0) \tag{62}$$

i.e., the solution starts with two trains of waves propagating in opposite directions,

In all the cases when the message ε depends on time, Eq. (35) remains linear (but with variable coefficients).

5. SYSTEMS DRIVEN BY STATE-DEPENDENT EVENTS

In this section we will discuss more complex structure of the input message ε assuming that it depends upon the state variable v:

$$\& = \varepsilon(v) \tag{63}$$

We will start with the simplest case when

$$\& = -\gamma^2 v, \gamma^2 << 1, \tag{64}$$

It can be verified by qualitative analysis of the system (compare with Eqs. (48) and (49)):

$$\dot{u} = y u^{1/3} \sin \omega t - \gamma^2 v \tag{65}$$

$$\dot{v} = \alpha u \tag{66}$$

that its solution will randomly oscillate about the point $v = 0$.

Indeed, when $v > 0$, then $\operatorname{sgn}\varepsilon < 0$, and v decreases; when $v < 0$, then $\operatorname{sgn}\varepsilon > 0$, and v increases. But when $v = 0$, then $\operatorname{sgn}\varepsilon = 0$, and the solution can escape the point $v = 0$ with the same probability 1/2 to the right or to the left.

The same result can be obtained by a formal analysis of the solutions (58), (59), and (62). Let us assume that the initial condition $v(t = 0)$ was set Up randomly with the probability distribution:

$$\Phi = \begin{cases} a & \text{if } |v| \leq \frac{1}{2a} \\ 0 & \text{otherwise} \end{cases} \tag{67}$$

in which h is given by Eq. (9), and ℓ is given by eq. (12).

Since the area enveloped by the function $\varphi(v,t)$ in (67) is shrinking, one have to turn to the normalized distribution (see Eq. (37):

$$\hat{\Phi}(v,t) = \begin{cases} \frac{a}{\frac{1}{2a} - h\ell} & \text{if } |v| \leq \frac{1}{2a} - h\ell \\ 0 & \text{otherwise} \end{cases} \tag{6s}$$

As follows from (68), in a finite time

$$T = \frac{\pi}{ah\omega} \tag{((N)}$$

the solution degenerates into a i-function.

This means that the solution will arrive? at the point $v = 0$ with the probability 1. However, at this point

$$\operatorname{sgn}\& = 0 \tag{70}$$

and Eq. (57) must be replace by Eq. (62) at n = 1

During the next time step the solution will be:

$$\hat{\Phi} = \begin{cases} \frac{1}{2}h\omega & \text{if } |v| \leq h\omega \\ 0 & \text{otherwise} \end{cases} \qquad \text{at } \overset{*}{t} = T + \frac{2\pi}{\omega} \tag{71}$$

This describes the onset of diffusion of the ii-function in both directions. However, for $t \geq \overset{*}{t}$, Eq. (62) must be replaced back by Eq. (57) since now sgn $\varepsilon \neq$ O, and the solution approaches the attractor v = O again as a 6-function. This periodic (in terms of probability) process corresponds to random oscillations of the dynamical system about the point v = O which qualitatively was described above.

In general case when $n > 1$, the behavior of the solution following from Eels. (58) and (64) remains qualitatively the same, with the only difference that the periodic behavior of the solution around the point v = O is replaced by multi - periodic one, while the largest amplitude of this oscillations:

$$A = nh\omega \tag{72}$$

Let us assume that instead of (64), the function

$$\& = \varepsilon(v) \tag{73}$$

has several zeros

$$\varepsilon(v_i) = 0, \text{and } \varepsilon(v_j) = \text{O}, \; i, j = 1,2, \text{“ “ “ etc.} \tag{74}$$

where

$$\frac{d\varepsilon}{dv}\Big|_{v=v_i} < 0, \text{ and } \frac{d\varepsilon}{dv}\Big|_{v=v_j} > 0 \tag{75}$$

Start with the case when the largest step in the generalized random walk is larger than the largest dist ante bet ween the neigl ıboring zeros v j:

$$nh\omega > \max_{j=1,2\ldots} | v_j - v_{j-1} | \tag{76}$$

Then the solution will eventually visit all the zeros v_i with the probability proportional to tile distance $|v_j - v_{j-1}|$ where

$$v_j < v_i < v_{j+1} \tag{77}$$

In the case when the largest step $nh\omega$ is less than some of the distances between the neighboring zeros $\overset{*}{v}_j$:

$$nh\omega < |\overset{*}{v}_j - \overset{*}{v}_{j-1}| \tag{78}$$

the solution will be trapped in the basins of these zeros v_i for which:

$$\overset{*}{v}_j < v_i < \overset{*}{v}_{j+1} \tag{79}$$

The zeros v_i can be considered as minima of the function:

$$\varepsilon = \int \varepsilon(v)dv \tag{80}$$

This means that the dynamical system (65) (66) can be exploited for finding local minima of an arbitrary function (80) such that these minima will be visited by the solution with probabilities proportional to the sizes of their basins.

That "rule" can be rearranged if one introduce in Eq. (13) the following change of variables:

$$y = z(1 + \beta^2\varepsilon^2), \ \beta = \text{const} \tag{81}$$

Then, with reference to Eqs. (15), and (18), one concludes, that in the dynamical system (65), (66), the largest step h of the generalized random walk will depend upon the depth of the minimum of the function (80):

$$H = nh\omega(1 + \beta^2\varepsilon^2), \ \beta \text{ const} \tag{82}$$

Hence, the solution of the dynamical system (65), (66) equipped with the additional rendition (81), will now visit the local minima of tile function (SO) with probabilities which are proportional to their depths as well -as the sizes of their 1 resins. In other wends, this dynamical system, with sufficiently large β, will find the global minimum of the fund ion (80).

More sophisticated "rules" of performance of the dynamical system (65), (66) can be implemented by changing the function φ in Eel. (13);

6. EVENTS DEPENDING UPON STATE VARIABLE PROBABILITIES

The complexit y of the dynamical systems considered above will be significantly enriched if events depend upon probabilities of the state variable v, and in particular, upon its statistical invariants such as expectation, variance, etc.

Starting with the dynamical system

$$\dot{u} = yu^{1/3} \sin \omega t + \varepsilon \tag{83}$$

$$\dot{v} = \alpha u \tag{84}$$

assume t hat

$$\varepsilon = -\gamma^2 \bar{v}, \ \gamma = \text{ const } \ll 1 \tag{85}$$

in which $\bar{v}$ is the mat hematical expect ation:

$$\bar{v} = \int_{-\infty}^{\infty} v\phi(v, t)dv \tag{86}$$

Two fundamental properties make the system (83), (84) different from all the previous Cases ,

Firstly, this system is coupled with the associated probability equation (35) since it contains the probabilit y distribution ϕ as a new unknown (see Eq. (86)).

Secondly, the probabilit y equation (35) becomes nonlinear since now the variable p explicitly depends upon the unknown distribution $\phi(v, t)$ (see Eqs. (32), (85) and (86)).

Nevertheless, Eqs. (83)-(W) are simple enough to be treated analytically. Indeed, consider the problem with the initial condition (67). As follows from the symmetry of (67) with respect to v = O:

$$\bar{v} = 0 \quad \text{at} \quad t = 0 \tag{87}$$

and t herefore

$$\text{sgn}\ \dot{\bar{v}} = 0 \quad \text{at} \quad t = 0 \tag{88}$$

The governing equation for the probability distribution for this case is obtained from (35) by salting

$$p = \frac{1}{2} \tag{89}$$

Hence, the solution to Eq. (35) starts from a symmetric diffusion, and therefore, the conditions (87), (S8), and (89) will persist. Eventually the solution will approach zero:

$$\phi(v, t) \to 0, \quad \bar{v} \to \infty \tag{90}$$

Thus, despite an apparent similarity between the dynamical systems (65), (66) and (83), (84), (85), their behaviors at the same initial conditions are significantly different.

As a second example, replace Eq. (85) by the following:

$$\dot{\bar{v}} = -\gamma^2[L\bar{v} - \overline{(v - \bar{v})^2}], \gamma \ll 1, L = \text{const} \tag{91}$$

where

$$\overline{(v - \bar{v})^2} = \text{Var}\ (\bar{v}) = \sigma^2 = \int_{-\infty}^{\infty} (v - \bar{v})^2 \phi(v, t) dv \tag{92}$$

and analyze the solution at the following initial conditions:

$$\phi(v, t) = \begin{cases} \frac{1}{L} & \text{if } 0 \le v \le L \\ 0 & \text{otherwise} \end{cases} \tag{93}$$

Since at $t = 0$

$$\bar{v} = \frac{L}{2}, \quad \text{Var}\ (\bar{v}) = \frac{L^2}{12} \tag{94}$$

and therefore

$$\operatorname{sgn} \varepsilon = -1 \tag{95}$$

the initial probability distribution (93), will shift to the left until

$$\operatorname{sgn} \& = 0 \tag{96}$$

For a simple random walk, i.e., when n = 1, the solution to Eq. (35) for $\phi(v,t)$ has the form of a single wave propagating without deformation (see Eq. (57)). In this case the state (96) can be found analytically.

Indeed, since the solution $\phi(v,t)$ can be represented by a moving rectangular, one. obtains:

$$\bar{v} = \frac{a_1 + a_2}{2}, \quad \operatorname{Var}(\bar{v}) = \frac{L^2}{12}, \quad L = a_1 - a_2 \tag{97}$$

where a_1 and a_2 are the coordinates at the right and the left ends of the rectangular, respectively.

Then the condition (96) occurs when

$$a_1 = \frac{7}{12} L \tag{98}$$

i.e. when

$$\phi(v,t) = \begin{cases} \frac{1}{L} & \text{if } -\frac{5}{12}L \le v \le \frac{7}{12}L \\ 0 & \text{otherwise} \end{cases} \quad \text{at } \overset{*}{t} = \frac{5\pi}{6} \frac{L}{h\omega^2} \tag{99}$$

The solution (99) is stable since

$$\frac{\partial^2 \varepsilon}{\partial a_1^2} < 0 \quad \text{and} \quad \frac{\partial \varepsilon}{\partial a_1} = 0 \quad \text{at} \quad \varepsilon = 0 \tag{loo}$$

and therefore, it will be valid for $t \ge \overset{*}{t}$.

Hence, the dynamical system *(83), (84),* (91) subject to the initial conditions (93) eventually approaches a stationary stochastic process with the probability distribution (99). However, this distribution depends upon the initial conditions (93) and therefore, it does not represent an attractor.

Such a simple analytical result could be obtained only for $n = 1$. If $n > 1$, the initial probability distribution changes its original shape.

7. MULTI-SCALE CHAINS OF EVENTS

In many problems of operation research, and especially, in decision analysis, one class of events can be much more important than snot]] er, so that in the presence of the first class of events ε_1, the second class ε_2 become decisive. In terms of the dynamical system (l), (8), (17), and (33), (46), this condition can be incorporated by modifying Eq. (46) as following:

$$\varepsilon = \delta\varepsilon_1 + \delta^2\varepsilon_2, \quad 0 < \delta \ll 1 \tag{101}$$

Here ε_1 and ε_2 can be considered as functions of time?, state variable and their probabilities, i.e.,

$$\varepsilon_1 = \varepsilon_1(t, v, \bar{v}, \cdots), \varepsilon_2 = \varepsilon_2(t, v, \bar{v}, \cdots) \tag{102}$$

Obviously, the second term in Eq. (101) can be ignored if $\varepsilon_1 \neq 0$ at the critical points (12), However, if $\varepsilon_1 = 0$, but $\varepsilon_2 \neq 0$ at these points, then the dynamical system is driven only by the event ε_2.

The evaluation of the probability distribution at v is described by the same equations (35), and (36), but Eq. (32) defining the probability p in there should be modified as following:

$$p = \begin{cases} 1 \text{ if } \operatorname{sgn}\varepsilon_1 = 1, \text{ or } \operatorname{sgn}\varepsilon_1 = 0, \text{ but } \operatorname{sgn}\varepsilon_2 = 1 \\ 0 \text{ if } \operatorname{sgn}\varepsilon_1 = -1, \text{ or } \operatorname{sgn}\varepsilon_1 = 0, \text{ but } \operatorname{sgn}\varepsilon_2 = -1 \\ \frac{1}{2} \text{ if } \operatorname{sgn}\varepsilon_1 = 0, \text{ and } \operatorname{sgn}\varepsilon_2 = 0 \end{cases} \tag{103}$$

In the same way one can introduce a multi-scale chain of events by modifying Eq. (101) as following:

$$\varepsilon = \delta\varepsilon_1 + \delta^2\varepsilon_2 + \cdots + \delta^n\varepsilon_n, \quad 0 < \delta \ll 1 \tag{104}$$

with the corresponding mollification of Eq. (104).

8. MULTI-DIMENSIONAL SYSTEMS

So far we were discussing the dynamical systems with only one state variable v (while u, x and y played the role of auxiliary variables). However, all the results obtained above can be generalized to dynamical systems which are characterized by the state variables $v_i, i = 1, 2 \cdots n$.

Let us start with Eqs. (1) and (8) and rewrite them in the following form:

$$\dot{x}_i = x_i^{1/3} \sin \omega t, \quad \dot{y}_i = x \quad : \tag{105}$$

As follows from (11), the probabilities $f_i = f_i(y_i, t)$ are governed by the difference equations:

$$f_i(y_i, t + \frac{2\pi}{\omega}) = \frac{1}{2} f_i(y_i - h, t) + \frac{1}{2} f_i(y_i + h, t), \int_{-\infty}^{\infty} f_i(y_i, t) dy_i = 1 \tag{106}$$

where h is expressed by Eq. (9).

By changing variables

$$z_i = \varphi_i(y_1, \cdots y_n), \quad y_i = \varphi_i^{-1}(z_1, \cdots z_n) \tag{107}$$

one can obtain a stochastic process with a prescribed probability distribution. Indeed, since y_i in (105) are statistically independent, the joint probability

$$f(y_1, y_2 \cdots y_n, t) = \Pi_{i=1}^{n} f_i(y_i, t) \tag{108}$$

and therefore, the? joint probability for z_i

$$\psi(z_1, \cdots z_n, t) = \Pi_{i=1}^{n} f_i[\varphi_i^{-1}(z_1, \cdots z_n)] \left| \det \frac{\partial y_i}{\partial z_j} \right| \tag{109}$$

Hence, the dynamical system (105), (107) characterized by the state variables $z_1, \cdots z_n$ performs a random motion with the joint probability function (109) found with the aid of the difference equations (106).

For better interpretation of Eq. (109), reduce Eqs. (107) to the. following parameterized form

$$z_i = \sigma\left(\sum_{i=1}^{n} T_{ij} y_j\right) \quad (7(") \quad \text{tanh}\ (\cdot), T_{ij} = \text{const} \tag{110}$$

Here $\sigma(\cdot)$ is a sigmoid function, while the representation (110) is "borrowed" from the neural net work architect ure.

Then

$$y_i = \sum_{i=1}^{n} T'_{ij} \sigma^{-1}(z_j), \sigma^{-1}(\cdot) = \text{Arctanh}\ (.) \tag{11"1}$$

and T'_{ij} are element S of the inverse matrix $\|T'_{ij}\|$

$$\|T'_{ij}\| = \|T_{ij}\|^{-1} \tag{112}$$

Since

$$\text{Det}\ \frac{\partial y_i}{\partial z_j} = \text{Det}\ \|\frac{T'_{ij}}{1 - z_j^2}\|, \quad -1 < z_j < 1 \tag{113}$$

one obtains instead of (109) :

$$\psi(z_1, \cdots z_n, t) = \Pi_{i=1}^{n} f_i \left[\sum_{i+1}^{n} T'_{ij} \sigma^{-1}(z_j)\right] \text{ det } \|\frac{T'_{ij}}{1 - z_j^2}\| \tag{114}$$

As follows from Eqs. (106), each variable y_i performs a simple symmetric unrestricted random work, and t herefore,

$$f_i(y_i) \to 0 \quad \text{at} \quad t \to \infty, \quad -\infty < y_i < \infty$$

However, as follows from Eq. (110),

$$-1 < z_i < 1 \tag{115}$$

and consequently,

$$\varphi(z_1, \cdots z_n, t) \to 0 \text{ at } t \to \infty \tag{116}$$

(Otherwise the condition $\int_{-\infty}^{\infty} \varphi(z_1, \ldots z_n, t) dz_1 \ldots dz_n \equiv 1$ cannot be enforced).

Thus, the solution to the dynamical system (105), (107) approaches a steady stochastic process (i. e., a stochastic attractor) with the joint probability expressed by Eq. (103) at $t \to \infty$:

$$\varphi_o(z_1 \ldots z_n) = \varphi(z_1 \cdots z_n, t) \text{ at } t \to \infty \tag{117}$$

Obviously φ_o is uniquely defined by the constant T_{ij} via Eqs. (111) and (1 13), and therefore, one can prescribe the joint probability φ_o by an appropriate choice of these constants.

Applications of stochastic attractors to information processing were discussed in [3,4].

Let us turn now to Eqs. (33) and (17) and generalize them to the following system:

$$\dot{u}_i = \varphi_i(y^{(i)}) u_i^{1/3} \sin \omega t + \varepsilon_i(t, v_i), 0 < \varepsilon_i \ll 1, \quad \varepsilon_i = \varepsilon_i(t, v_i, \dot{v}_i \cdots) \tag{118}$$

$$\dot{v}_i = \alpha u_i \tag{119}$$

where α is expressed by Eq. (17),

Then one arrives at the difference equations for the probabilities ϕ_i $(v_i t)$ similar to eqs. (35), (30):

$$\phi_i(v_i, t + \frac{2\pi}{\omega}) = \sum_{k=1}^{n} p_k^{(i)} \{p_i \phi_i[v_i - \varphi_i(kh)] + (1 - p_i)\phi_i[v_i - \varphi(kh)]\} \tag{120}$$

Here, with reference to Eq. (32):

$$p_i = \begin{cases} 1 & \text{if } \operatorname{sgn} \varepsilon_i = 1 \\ 0 & \operatorname{sgn} \varepsilon_i = -1 \\ \frac{1}{2} & \text{if } \varepsilon = 0 \end{cases} \tag{121}$$

and with reference to Eq. (23):

$$p_k^{(i)} = f_i[\varphi_i^{-1}(kh, t)] \mid \frac{d\varphi^{-1}}{dz_i} \mid, \quad z_i = \varphi_i(y^{(i)}) \tag{122}$$

where f_i is defined by Eq. (106).

So far the variables v_i in Eqs. (1 13) and (119), as well as the probabilities ϕ_i in Eq. (120), are independent. That is why the joint probability $\phi(v_1, \ldots v_n, t)$ can be found as:

$$\phi(v_1, \cdots v_n, t) = \Pi_{i-1}^{n} \phi_i(v_i, t) \tag{123}$$

Let us change variables v_i as following:

$$w_i = \sigma(\sum_{i=1}^{n} T_{ij} v_j), \quad \text{where} \quad v_i = \sum_{i=1}^{n} T'_{ij} \sigma^{-1}(w_j) \tag{124}$$

Here $\sigma, \sigma^{-1}, T_{ij}$ and T'_{ij} are defined by eqs. (110), (111) and (1 12).

Then the dynamical system (1 18), (1 19) is expressed via the new variables w_i:

$$\dot{u}_i = \varphi_i(y^{(i)}) u_i^{1/3} \sin \omega t + \varepsilon_i[t, v_i(w_1, \cdots w_n)] \tag{125}$$

$$\dot{w}_i = \alpha(1 - w_i^2) \sum_{i=1}^{2} T_{ij} u_j \tag{126}$$

The joint probability $\theta(w_1, \cdots w_n, t)$ is found from Eq. (123) by formal change of variables:

$$\theta(w_1, \cdots w_n, t) = \phi[v_1(w_1, \ldots w_n), \cdots v_n(w_1, \cdots w_n)] \operatorname{Det} \| \frac{T'_{ij}}{1 - w_j^2} \| \tag{127}$$

where $v_i(w_1, \cdots w_n,)$ are given by Eqs. (124), while $\phi_i(v_i, t)$ are defined by Eqs. (120).

Thus, the dynamical system (125), (126) represents an n-dimensional generalized non-symmetric random walk.

After each time period $\tau = 2\pi/w$, all the variables w_i change their values on

$$\triangle w_i = (\text{ sgn } \varepsilon_i)\varphi_i(h) \quad \text{if} \quad \varepsilon_i \neq 0, \qquad (128)$$

where $\varphi_i(h)$ are random variables: their probability densities follows from Eqs. (106) and (122).

The sizes of the steps (12S) can be correlated if one introduces the following constraints:

$$\sum_{i=1}^{m} \Omega_{ij}\varphi_j(y^{(j)}) \cdot 0, \; \Omega_{ij} = \text{const} \; , \; m \; 7 \; 1 \; , \qquad (129)$$

where, for the concretene ss, $\varphi_i(\cdot)$ can be chosen as:

$$\varphi_i(\cdot) \; = \tanh(\cdot) \qquad (130)$$

By appropriate? choice Of the coefficients Ω_{ij}, one can create more and less preferable transitions of the dynamical system from one state to another.

The directions of the steps (12S) are governed by the signs of the events &i. Therefore, according to Eq. (125), they depend upon the time t and the st ate variables w_i. In general case they can also depend upon the statistical invariant $\bar{w}_i, \bar{w}_i^2$ etc.

Special attention should be paid to the branching points at which

$$\varepsilon_i = 0 \qquad (131)$$

At these points the direction of the next step is not defined:

$$\triangle w_i = \pm\varphi_i(h), \qquad (132)$$

so with an equal probability 1/2 the variable w_i can move in positive or negative directions.

From the viewpoint of information processing, the. branching points are very important: they incorporate the probability of "sudden" changes in the behavior of the dynamical system. Obviously, the location of these points as well as tile domains of positive and negative ε_i can be uniquely prescribed by the constants T_{ij}.

Thus, the. behavior of the dynamical system (125), (126) is uniquely defined by tile joint probability evolution (127), and it can be prescribed by the appropriate choice of the constants T_{ij} and Ω_{ij}.

It should be emphasized that the variables w_i in the system (125), (126) are coupled in two different ways: the coefficients Ω_{ij} provide statistical correlation between the sizes of steps Δw_i, while the coefficients T_{ij} correlate the state-clc~l]cllclcllt events ε_i which are responsible for dire ctions in which the dynamical system moves.

Hence, even a brief analysis of the performance of the dynamical system (125),(126) demonstrates that the complexity of its behavior matches the complexity of behavior of typical {list.r~?te-c?vc?llts systems which occur in information processing structures, in social dynamics, in decision making processes, etc. At the same time, this dynamical system possesses a relatively simple and fully tractable an al ytical st ructure which allows one to analyze it not only numerically, but qualitatively as well.

9. SYNTHESIS OF DISCRETE-EVENTS SYSTEMS

So far our attention was focused on analysis of terminal models for discrete events dynamics. In this section we will draft possible approaches to synthesis of these systems. Since? the discrete-event dynamical systems discussed above, are uniquely defined by the constant parameters T_{ij} and Ω_{ij} (in the sense that these parameters uniquely define the evolution of joint probability of the state variables, given by Eq. (127)), the problem of the synthesis can be reduced to finding the parameters T_{ij} and Ω_{ij} in such a way that the objective of the performance is achieved.

We will consider four problems of the synthesis associated with systems identification, optimization based upon global objective, optimizat ion based upon local rules, and systems with collective brain.

a. System Identification

The problem of system identification arises when the anal ytical structure of the dynamical process performed by the system is unknown. Then, based upon experimental data, a phenomenological version of the dynamical system wh ich has an identical input-output characteristics is developed. For deterministic systems, the process of parameter identification reduces to a nonlinear optimization problem. The same approach can be formally applied to a discrete-event system if the objective is to reproduce its behavior in terms of state variable probability evolution. Indeed, in this case one can turn to Eq.

(127) which uniquely defines this evolution in terms of the parameters T_{ij} and Ω_{ij} and solve the inverse problem of finding these parameters from given input-output data [2].

Recently, along with the formal mathematical approach to system identification, several biologically inspired methods borrowed from tile neural networks theory were developed. In connection with the discrete event systems, the strategy for application of these methods may be the following.

Let us assume that experimentally observed behavior of the system can be statistically approximated by a histogram which describes the distribution of frequency with which certain states are visited by the dynamical system. Such an approximation is biologically meaningful since the frequency mentioned above is proportional to the strength of the memory trace for the corresponding pattern of behavior.

If the experimental histogram is presented as

$$\hat{\phi} = \hat{\phi}(w_1, \cdots w_n) \tag{133}$$

then, int reducing the "energy" function:

$$E = \int_{w_1, \cdots w_n} [\hat{\phi}(w_1, \cdots w_n) - \phi(w_1, \cdots w_n, T_{ij}, \Omega_{ij})]^2 dw_1, \cdots dw_n \rightarrow \min \tag{134}$$

one can derive the following learning dynamics:

$$\dot{T}_{ij} = -\frac{\partial E}{\partial T_{ij}}, \ \dot{\Omega}_{ij} = -\frac{\partial E}{\partial \Omega_{ij}} \tag{135}$$

where $\phi(w_1, " " " w_n, T_{ij}, \Omega_{ij})$ can be found as a solution to Eq. (127), or it can be reproduced by the dynamical system (125), (126) for each particular $\overset{*}{T}_{ij}$ and $\overset{*}{\Omega}_{ij}$.

The system (135) will converge to a minimum (which will be a global minimum if ϕ is a quadratic form of T_{ij}, and Ω_{ij}), since E plays the role of the Lyapunov function.

b. Optimization Based Upon Global Objective

In many problems of operation research, the objective of the performance of a discrete event system is to minimize expectations of a certain combination of state variables with possible const mints imposed upon other statistical invariants, for instant:

$$E = \theta(\bar{w}_1, \cdots \bar{w}_n) \rightarrow \min, \tag{136}$$

or

$$E = \int_{t_1}^{t_2} \theta(\bar{w}_1, \cdots \bar{w}_n, t)dt \rightarrow \min, \tag{137}$$

while

$$\bar{w}_i^2 \leq \overset{*}{w}_i^2 = \text{ const} \tag{138}$$

Clearly the solution to the problem of finding the optimal values of T_{ij} and Ω_{ij} for the dynamical system (125), (126) minimizing the performance indices (136), or (137) can be reduced to the case (134), (135) considered above.

c. Optimization Based Upon Local Rules

In many real-life. situations, a member of a biological, or a social system does not have an explicitly formulated global objective for the whole system. Instead, it has its own local objective which can be partly compatible with, and partly contradictory to the local objet.tivcs of other members. In addition to that, each member may try to copy the behavior of a "successful" neighbor, or a leader, based upon local rules, and these rules couple the evolut ion of all the members of the system. Eventually such a system may approach a state which can be interpreted as the global objective of the performance.

Let us turn to analytical formulations of the local rules.

The local objectives of each member (or, a dynamical unit) can be introduced by reducing Eq. (136) to the following:

$$E_i = E_i(w_i, T_{ii}, \Omega_{ii}) \rightarrow \min; \tag{139}$$

where

$$\dot{T}_{ii} = -\frac{\partial E_i}{\partial T_{ii}}, \dot{\Omega}_{ii} = -\frac{\partial E_i}{\partial \Omega_{ii}}, \quad i = 1, 2 \cdots n \tag{140}$$

Clearly, Eq. (140) defines only the diagonal elements "of the matrices T_{ij} and Ω_{ij}.

In order to define the non-diagonal elements of these matrices, first we will assume that the indeces i and j Of the elements T_{ij} and Ω_{ij} are related to spatial locations (cm coordinates) of the corresponding elements so that the positive integer $|i - j|$ is proportional to the spat.ial distance between the dynamical units characterized by the variables w_i and w_j. Then, it can be assumed that T_{ij} is inversely proportional to the distance $|i$ -j $|, (i \# j)$. Indeed, it would mean that the close. neighbors effect each other behavior more strongly than the more distant ones.

However, the spacial distance between the dynamical units is not the only measure of the degree of interaction between them: the distance in the functional space may occur to be even more important. Such a distance between the units i and j can be introduced as the scalar $|w_i - w_j|$. Then the local rule for the non-diagonal elements can be formulated, for instance, as following:

$$T_{ij} = \frac{\beta}{|i - j|} e^{-\alpha(w_i - w_j)^2} = T_{ji}, \ (i \neq j) \tag{141}$$

where α is a constant of the same dimensionality as $1/w^2$, and β is the constant. The coefficients $\Omega_{ij(i \neq j)}$ can be defined in a similar way.

As follows from Eq. (141), the interaction between two dynamical units increases with the decrease of both the spatial distance $|i - j|$ and a functional distance $|w_i - w_j|$. In simple words, it means that the strongest interaction occurs between close neighbors who are in the same "income" bracket.

As a result of the local rules (140), and (141), the dynamical system (125), (126) will eventually approach some stochastic process which can be associated with a certain optimization problem defined implicitly via these rules. However, in general it is a very difficult (if not an impossible) task to reconstruct the global objective of the system performance based only upon the local rules, but without an actual run of the system.

cl. Systems with Collective Brain

The concept of the collective brain has appeared recently as a subject of intensive scientific discussions from theological, biological, ecological, social, and mathematical viewpoint [4]. It can be introduced as a set of simple units of intelligence (say, neurons) which can communicate by exchange of information without explicit global control. The objective of each unit may be partly compatible and partly contradictory, i.e., the units can cooperate or compete. The exchanging information may be at times inconsistent, often imperfect, non-deterministic, and delayed. Nevertheless, observations of working insect colonies, social systems, and scientific communities suggest that such collectives of single units appear to be very successful in achieving global objectives, as well as in learning, memorizing, generalizing and predicting, due to their flexibility, adaptability to environmental changes, and creativity.

In [4] collective? activities of a set of unites of intelligence were represented by a dynamical system which imposed upon its variables different types of non-rigid constraints such as probabilistic correlations via the joint density. It was assumed that these probabilistic correlations are learned during a long-term period of performing collective tasks. Due to such correlat ions, each unit can predict (at least, in terms of expect at ions) the values of parameters characterizing the activities of its neighbors if the direct exchange of information is not available. Therefore, a set of units of intelligence possessing a "(knowledge base" in the form of joint density function, is capable of performing collect ive purposeful tasks in the course of which the lack of information about current states of units is compensated by the predicted values characterizing these states. This means that actually in the collective brain, global control is replaced by the probabilistic. correlations bet ween the units stored in the joint density functions.

In the framework of the discrete events dynamics considered above, the collective brain paradigm can be incorporated in the. following way.

Let us assume that each member (or a dynamical unit) characterized by the variable w_i, has its own version of the global objective of tile whole dynamical system which can be expressed in the form similar to (136):

$$E_i = \theta_i(\bar{w}_1, \cdots \bar{w}_n) \to \min \tag{142}$$

while, in general,

$$E_i \neq E_j \quad \text{if} \quad i \neq j \tag{143}$$

Each unit can learn (in its own way) the global objective of the system during previous collect ive tasks. Based upon that, it may "derive" its own version of the learning dynamics similar to (135):

$$\dot{T}_{ij}^{(k)} = -\frac{\partial E_k}{\partial T_{ij}}, \dot{\Omega}_{ij}^{(k)} = -\frac{\partial E_k}{\partial \Omega_{ij}}, \quad k = 1, 2, \cdots n \tag{144}$$

and therefore, its own version of the whole. dynamical system in the form similar to Eqs. (125) and (126):

$$\dot{u}_i^{(k)} = \varphi_i(y_1^{(i)}) u_i^{1/3} \cos \omega t + \varepsilon_i[t, v_i(w_1, \cdots w_n, w_1^{(k)}, \cdots w_n^{(k)})] \tag{145}$$

$$w_i^{(k)} = \alpha[1 - (w_i^{(k)})^2] \sum_{i=1}^{n} T_{ij}^{(k)} u_j^{(k)}, \quad k = 1, 2, \cdots n$$

Here $u_i, w_1, \cdot . \cdot w_n$ are the actual values of the variables characterized the dynamical systems, and $u_i^{(k)}, w_i^{(k)}, \cdot \cdot w_n$ are the. values of the same variables predicted by the kth dynamical unit, while $u_i^{(i)} = u_i, w_i^{(i)} = w_i$.

It is assumed that in the course of performance of the original dynamical system **(125), (126),** not all the actual values of the variables u_i and w_i are available. In this case, the unavailable? variables are replaced by their predict ed values, while each dynamical unit predicts them based upon its own version of the dynamical system (see Eq. (145).

Hence, as a result of the collective? brain paradigm, the original dynamical system (125), (126) of $2n$ equations with respect to $2n$ variables: u_i and W: is replaced by t he system (145) of $2n^2$ equations with respect to $2n^2$ variables: $u_i^{(k)}$ and $w_i^{(k)}$ (i = 1,2, 00071, k = $1, 2 \cdots n$).

Since the last system has the same dynamical structure as the original dynamical system (125), **(126),** its solution can be described by n^2-dimensional joint probability similar to (127):

$$\theta = \theta(w_1, \cdots w_n, w_1^{(1)} \cdots w_n^{(1)} \cdots w_n^{(1)} \cdots w_n^{(n)}) \tag{146}$$

As follows from Eq. (146), the dynamics with the collective brain is less predictable than the original dynamics. However, in contradist inction to the original dynamics which requires a global control for its performance, the last version of dynamics is more flexible: it can perform relatively well based upon t he autonomy of t he dynamical units which can predict the events if the actual values of the variables are not available.

The autonomy of the dynamics with collective brain can be increased if each unit can have not only its own version of the global objective of the system, but also its versions of the global objectives of others dynamical uni ts. Clearly such an ability will require deeper correlations between the? d ynamical units whi ch can be achieved by more intensive learning during the? previous collective tasks. From the analytical viewpoint, the complexity of this dynamical system will be significantly higher: the system having the same structure as Eqs. (125), (126), or (144), (145), will contain $2n^3$ equations with respect to $2n^3$ variables. In the same way one can introduce more autonomous (but more complex) dynamical systems with collective? brain of higher dimensionalities.

The discrete event dynamics of the type (145), or of its more complex versions mentioned above, can be linked to game? theory. Indeed here each ith player (represented by the corresponding variable w_i) tries to achieve its local objective by taking into account the knowledge about possible. local objectives of another players. However, in contradistinction to the classical game theory which can be associated with artificial intelligence (since it is based upon sets of rules and strategies), the discrete event c1 ynamics version discussed above, can be rather associated with neural networks: it is represented by a dynamical system, and the knowledge is acquired and stored in the coefficients T_{ij} and Ωij in the course of learning.

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